Using Polynomials as a Form-Fitting Tool.

By Dirk Mittler, August 16, 2013

As an example, a cubic (3rd-order) polynomial can be given as follows:

$$f(x)=a1\ x^3+a2\ x^2+a3\ x+a4.$$

This type of function can be made to give a list of 4 values, for a list of 4 known parameters (x), through a careful choice of a1 ... a4 :

$$a1\ x_1^3+a2\ x_1^2+a3\ x_1+a4=f(x_1).$$
$$a1\ x_2^3+a2\ x_2^2+a3\ x_2+a4=f(x_2).$$
$$a1\ x_3^3+a2\ x_3^2+a3\ x_3+a4=f(x_3).$$
$$a1\ x_4^3+a2\ x_4^2+a3\ x_4+a4=f(x_4).$$

Let $A=\begin{pmatrix} a1 \\ a2 \\ a3 \\ a4 \end{pmatrix}$ and Let $Y=\begin{pmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{pmatrix}$.

And Let X= $\begin{pmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ x_2^3 & x_2^2 & x_2 & 1 \\ x_3^3 & x_3^2 & x_3 & 1 \\ x_4^3 & x_4^2 & x_4 & 1 \end{pmatrix}$ [ From known values of (x) ].

Then, $XA = Y$.

But, we're assuming that Y and X are known, while A needs to be found.
Thus, we need to find the inverse of X, such that

$$X^{-1} Y = A.$$

Now, the polynomial which results, will fit 4 exact pairs of (x) and f(x).
But one side-effect of doing this, is that the positions of the local maxima and minima are unpredictable. In fact, local maxima and minima usually occur where $f'(x)=0$.
Therefore, we'd like to know the trick by which instead of 4 points, the polynomial can be made to fit 2 points and 2 derivatives, or 3 points and 1 derivative, or 1 point and 3 derivatives, etc...

$$f'(x)=3\,a1\,x^2+2\,a2\,x+1\,a3+0\,a4.$$

Thus, we can Let Y = $\begin{pmatrix} f(x_1) \\ f(x_2) \\ f'(x_1) \\ f'(x_2) \end{pmatrix}$.

And Let X= $\begin{pmatrix} x_1^3 & x_1^2 & x_1 & 1 \\ x_2^3 & x_2^2 & x_2 & 1 \\ 3\,x_1^2 & 2\,x_1 & 1 & 0 \\ 3\,x_2^2 & 2\,x_2 & 1 & 0 \end{pmatrix}$.

Thus, if we simply wanted to solve the 1-dimensional cubic interpolation between x = 0 and x = 1, then we could use:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{pmatrix} A = \begin{pmatrix} f(0) \\ f(1) \\ f'(0) \\ f'(1) \end{pmatrix}.$$

Yet, it becomes easy to see that larger form-fitting exercises are possible, with polynomials up to the (n)th-degree.

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