This worksheet is meant to exploare what happens, if we raise a complex base, first to a rational power, and if we next contemplate an irrational power. Assumption: Complex Numbers can represent both Cartesian and Polar, coordinate systems. The logartithm of a complex number has as real part, the log of the absolute of the parameter, and as repeating imaginary part, the polar angle of the complex parameter. For the moment, I'll just get a definition taken care of, that I'll need later on.

(%i1) ComplexToList(C) := [trigreduce(realpart(rhs(C))), trigreduce(imagpart(rhs(C)))]$

Hint: To find the cube root of a complex number, also means to raise to the power of 1/3, but really asks the question, 'What complex numbers may I raise to the power of 3, to arrive at my parameter?'

(%i2) Set: solve(xˆ3=1,x);

$$[x = \frac{\sqrt{3}\%i - 1}{2}, x = -\frac{\sqrt{3}\%i + 1}{2}, x = 1] \tag{Set}$$

The following List (L), will contain the Real and Imaginary parts, of the solution-set (Set), which is encumbered by the equals sign.

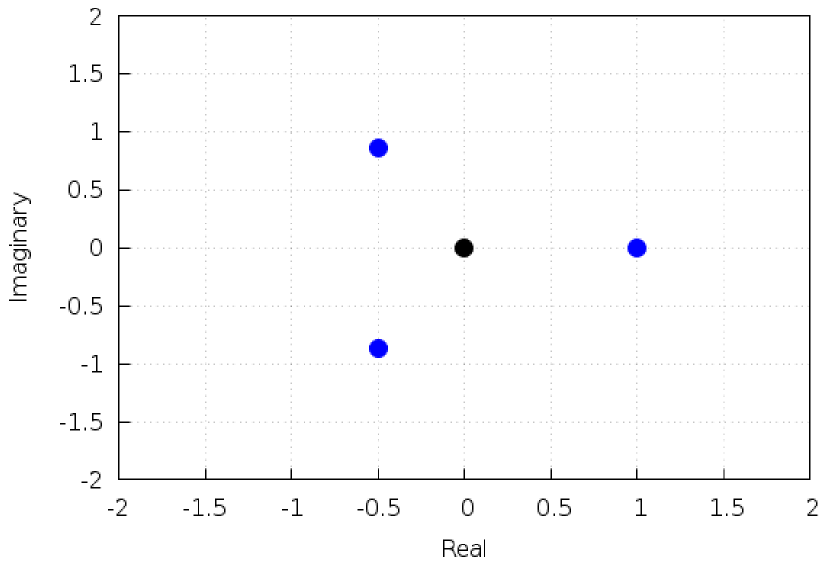
(%i3) L: makelist (ComplexToList(x), x, Set);

$$[[-\frac{1}{2}, \frac{\sqrt{3}}{2}], [-\frac{1}{2}, -\frac{\sqrt{3}}{2}], [1, 0]] \tag{L}$$

(%i4) load("draw")$

;; Loading "/home/dirk/.maxima/binary/5_38_1/gcl/GCL_2_6_12/share/draw/grcommon.o";; Finished loa

(%i5) wxdraw2d(point_size=2,
point_type=filled_circle,
color=blue,
xlabel="Real",
ylabel="Imaginary",
grid=true,
xrange=[-2,2],
yrange=[-2,2],
points(L),
color=black,
points([[0,0]]))$



(%t5)

The logarithm-function will now be used to extract the polar angle of the points above, but converted into degrees.

(%i6) OutputPolar(L) := makelist([abs(rhs(x)), imagpart(log(rhs(x)))*180/%pi], x, L)$

(%i7) OutputPolar(Set);

$$[[1, 120], [1, -120], [1, 0]] \quad (\%o7)$$

The original definition of the cube root also asks the question, 'What polar angle may I multiply by 3, to arrive at a mutliple of 360 degrees, when in a circle, any mutliple of 360 degrees is equivalent?' Now, the same exercise can be repeated, but in order to analyze 1 ^ (1/5) .

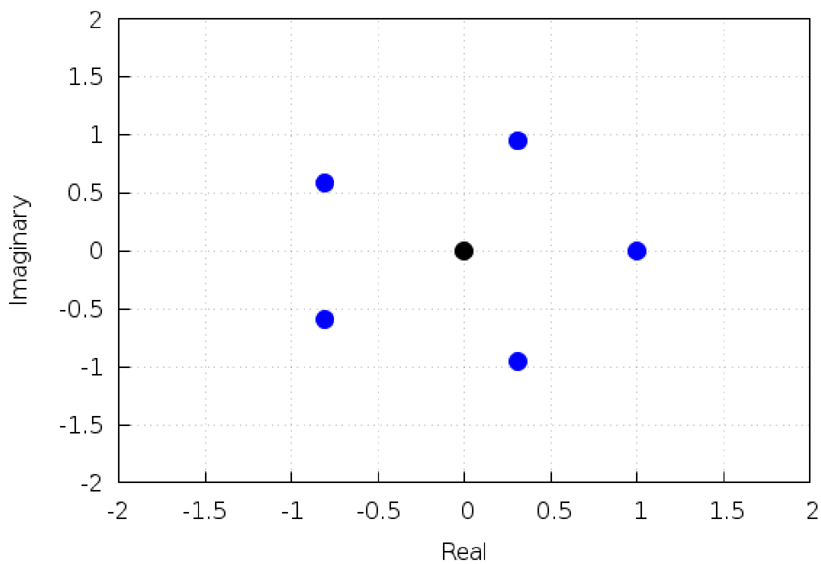
(%i8) Set: solve(xˆ5=1, x);

$$[x = \%e^{\frac{2\%i\pi}{5}}, x = \%e^{\frac{4\%i\pi}{5}}, x = \%e^{-\frac{4\%i\pi}{5}}, x = \%e^{-\frac{2\%i\pi}{5}}, x = 1] \quad \text{(Set)}$$

(%i9) "%i3;

$$[[\cos\left(\frac{2\pi}{5}\right), \sin\left(\frac{2\pi}{5}\right)], [\cos\left(\frac{4\pi}{5}\right), \sin\left(\frac{4\pi}{5}\right)], [\cos\left(\frac{4\pi}{5}\right), -\sin\left(\frac{4\pi}{5}\right)], [\cos\left(\frac{2\pi}{5}\right), -\sin\left(\frac{2\pi}{5}\right)], [1, 0]] \quad \text{(\%o9)}$$

(%i10) "%i5;



(%t10)

(%o10)

(%i11) "%i7;

$$[[1, 72], [1, 144], [1, -144], [1, -72], [1, 0]] \quad \text{(\%o11)}$$

Answer: '72 degrees, or 144, may be multiplied by 5, to arrive at multiples of 360 degrees.'

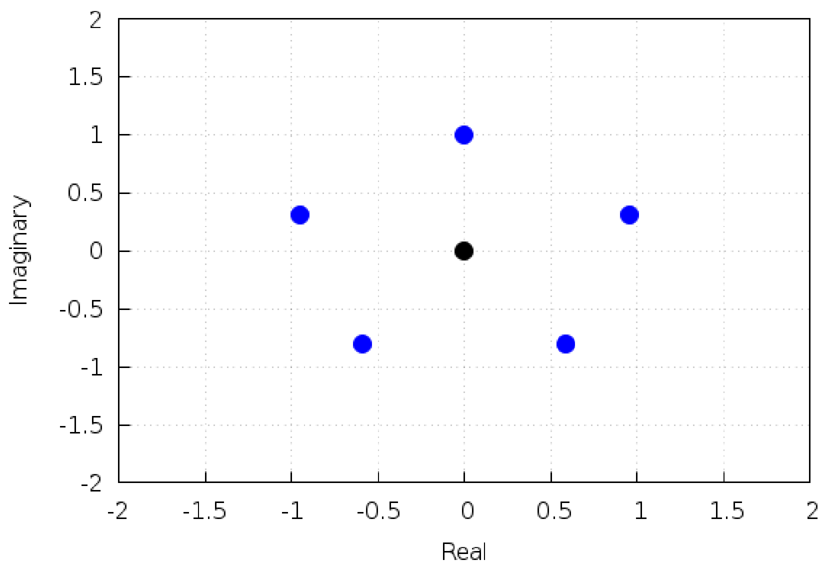
(%i12) Set: solve(xˆ5 = %i, x);

$$[x = (-1)^{\frac{1}{10}}\,\%e^{\frac{2\%i\pi}{5}}, x = (-1)^{\frac{1}{10}}\,\%e^{\frac{4\%i\pi}{5}}, x = (-1)^{\frac{1}{10}}\,\%e^{-\frac{4\%i\pi}{5}}, x = (-1)^{\frac{1}{10}}\,\%e^{-\frac{2\%i\pi}{5}}, x = (-1)^{\frac{1}{10}}] \quad \text{(Set)}$$

(%i13) ''%i3;

$$[[0,1],[\cos\left(\frac{9\pi}{10}\right),\sin\left(\frac{9\pi}{10}\right)],[\cos\left(\frac{7\pi}{10}\right),-\sin\left(\frac{7\pi}{10}\right)],[\cos\left(\frac{3\pi}{10}\right),-\sin\left(\frac{3\pi}{10}\right)],[\cos\left(\frac{\pi}{10}\right),\sin\left(\frac{\pi}{10}\right)]] \qquad (\%o13)$$

(%i14) ''%i5;



(%t14)

(%o14)

(%i15) float(''%i7);

$$[[1.0,90.0],[1.0,-198.0],[1.0,-126.0],[1.0,-54.0],[1.0,18.0]] \qquad (\%o15)$$

Hence: What would happen if we raised a complex base, to a fraction in 1/10ths? Or to a fraction in 1/20ths? We'd obtain 10 roots, or 20, respectively. But, irrational numbers can be seen as though fractions, with infinite denominators. Hence, if we used those as exponents, we'd be implying the existence, of an infinite number of solutions !

(%i16) solve(x^(sqrt(2)) = 1, x);

$$[x=1] \qquad (\%o16)$$

Conclusion: By convention, if a solver has an infinite number of roots to answer with, it will answer with the first root that it finds, and, if neither the base of an exponentiation nor its power was explicitly complex, the assumption will follow that we only want real (non-complex) solutions.

(%i17) solve(xˆ(sqrt(2)) = 2, x);

$$[x = 2^{\frac{1}{\sqrt{2}}}] \tag{\%o17}$$

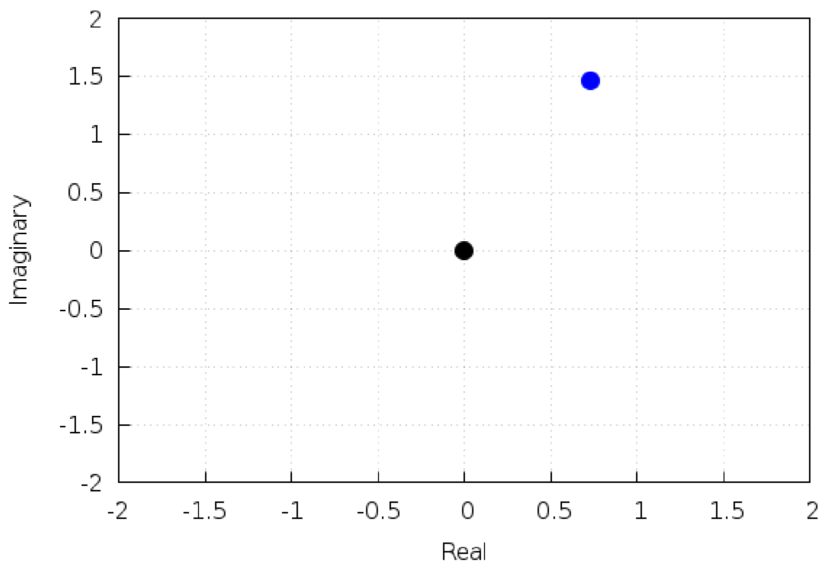
(%i18) Set: solve(xˆ (sqrt(2)) = 2 * %i, x);

$$[x = (-1)^{\frac{1}{2^{\frac{3}{2}}}}\, 2^{\frac{1}{\sqrt{2}}}] \tag{Set}$$

(%i19) "%i3;

$$[[2^{\frac{1}{\sqrt{2}}} \cos\left(\frac{\pi}{2^{\frac{3}{2}}}\right), 2^{\frac{1}{\sqrt{2}}} \sin\left(\frac{\pi}{2^{\frac{3}{2}}}\right)]] \tag{\%o19}$$

(%i20) "%i5;



(%t20)

(%o20)

(%i21) "%i7;

$$[[2^{\frac{1}{\sqrt{2}}}, 45\sqrt{2}]] \tag{\%o21}$$

Answer: The second element of the pair above may be multiplied by the square root of (2), to arrive at an angle of 90 degrees, which was also the polar angle of the base. The first element of the pair reflects the absolute of the root.

(%i22) AngleBetweenRoots: 2 * %pi / sqrt(2)$

(%i23) Phasor: exp(%i * AngleBetweenRoots);

$$\%e^{\sqrt{2}\%i\pi} \quad \text{(Phasor)}$$

(%i24) [Phasor^sqrt(2), Phasor^(2 * sqrt(2)), Phasor^(3 * sqrt(2))];

$$[1, 1, 1] \quad (\%o24)$$

(%i25) FirstRoot: float(rectform(rhs(Set[1])))$

(%i26) makelist(float(rectform((x)^sqrt(2))), x, [FirstRoot,
FirstRoot * Phasor,
FirstRoot * (Phasor^2)]);

$$[2.0\%i{+}5.66553889764798 10^{-16}, 2.0\%i{-}1.164033439826574 10^{-15}, 1.026576794314127{-}1.716432371337632\%i] \quad (\%o26)$$

Note: Amounts with an order of magnitude of 10^-15 or smaller, are round-off errors. If I try to multiply (FirstRoot) by (Phasor) to some power, I should get the same result again, but do not. The reason is the fact, that when the above line raises the product to the power of (sqrt(2)) a second time, after (Set) was set, that exponentiation chooses a root which is inconvenient for me.

(%i27) abs((%o26)[3]);

$$2.0 \quad (\%o27)$$

(%i28) (2^3)^4;

$$4096 \quad (\%o28)$$

(%i29) 2^(3 * 4);

$$4096 \quad (\%o29)$$

Conclusion: Mathematics is already built on the theory, that if we take an arbitrary real number, and start to fill a number-line with only the rational products of that real number, 'The number line is not completely filled.' And this is thought to be true, even though our set at that point has an infinite number of members. This is also the basis for saying, that the set of rational numbers, does not include all real numbers. We could be trying to fill the number line with all rational products of 2 * Pi, and there would still be values on the number line, which do not get mentioned. Therefore, in the above example, we could have a set of complex numbers, all of which have as absolute (2), but which have as a polar-coordinate angle, all the rational products of ((x * 2 * Pi / sqrt(2)) - (y * 2 * Pi)) , where (x) and (y) are non-zero integers. If in addition to that we chose a polar angle which was random, we would not be hitting any of the true members of this set. And so, any polar coordinate would not be a correct answer, to what the root was. Only certain, exact polar coordinate- pairs, would count as true answers, to what this root was.