How the basic chain rule applies to integrals.

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One concept which was first taught in calculus 1 was, that the way derivatives are computed, is subject to a "Chain Rule". What this states is that, when computing the derivative of a function, which is defined as feeding its parameter to a nested function, after multiplying that parameter by (a), then the derivative of the nested function must also be multiplied by (a), in order to compute the derivative of the principal function. Obviously, this can also be stated purely Mathematically, and this is how to do so:

$F(x) = \sin(ax)$
$t = ax$
$F(x) = \sin(t)$
$\frac{dF(x)}{dx} = \frac{d\sin(t)}{dt}\frac{dt}{dx}$
$\frac{dt}{dx} = a$
$\frac{d\sin(t)}{dt} = \cos(t)$
$\therefore \frac{dF(x)}{dx} = a\cos(ax)$

When computing integrals, chain rule has the opposite effect. The integral of the nested function needs to be divided by (a), in order to compute the integral of the principal function:

$G(x,a) = \cos(ax)$
$H(a) = \int\limits_{x=0}^{1} G(x,a)\,dx$
$= \int\limits_{x=0}^{1} \cos(ax)\,dx$
$t = ax$
$\frac{dt}{dx} = a$
$(dx) = \frac{dt}{a}$
$H(a) = \int\limits_{x=0}^{1} \cos(t)\,dx$
$= \int\limits_{t=0}^{a} \frac{\cos(t)}{a}dt$
$\therefore H(a) = \left[\frac{\sin(t)}{a}\right]_0^a = \left(\frac{\sin(a)}{a} - \frac{\sin(0)}{a}\right) = \frac{\sin(a)}{a}$

Hence, this is where the definition of the sinc function originates, which is:

$sinc(a) \equiv \frac{\sin(a)}{a}$

A way to visualize this would be, to assume that the sine function of a multiplier of (x) is to be plotted, over the interval $x \in [0, 1]$, but in such a way that the sine function completes a half-cycle, which it would do from $t \in [0, \pi]$. Thus, if (a=π), then this would be the plot:

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F(x) = sin( pi * x )
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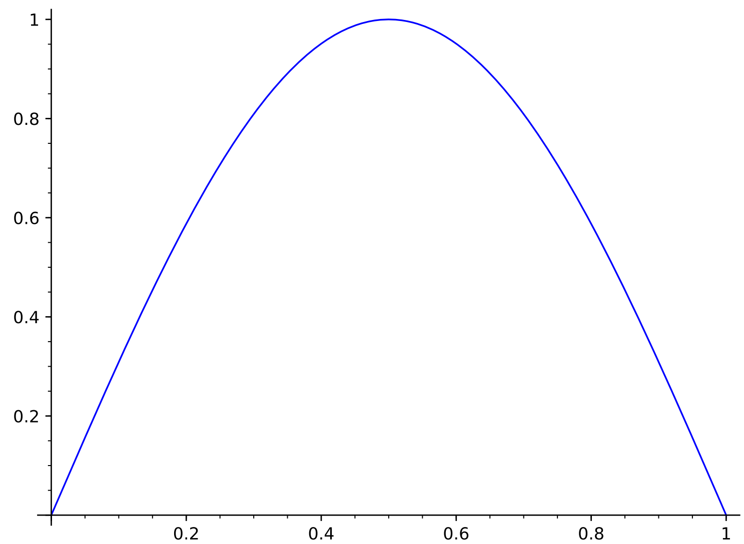

The Area (A) under this curve, will be...

$$\left[\frac{-\cos(\pi x)}{\pi}\right]_0^1 = \frac{2}{\pi}$$

-by Dirk Mittler