The purpose of this worksheet is, to explore the question of whether 'tripled roots' of polynomials are, in fact, a real possibility, which would also be known as a 'multiplicity of 3'.

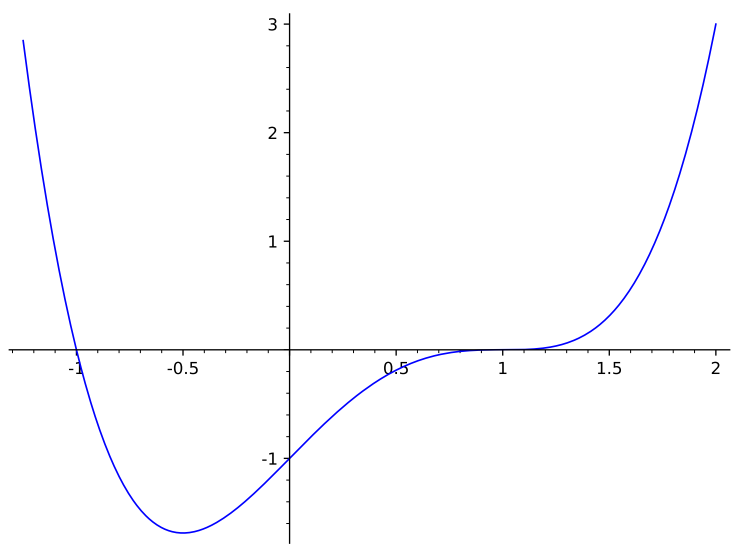
The most trivial example might be:
xˆ3 = 0

But, because we cannot see the forest for the trees, I'll just assume that this example is insufficient. In order to start, a quartic polynomial can be constructed, which has (+1) with a mutliplitiy of 3, but which only has (-1) as a single real root...

```
var('x')
F(x) = expand((x - 1)*(x - 1)*(x - 1)*(x + 1))
```

$$x^4 - 2\,x^3 + 2\,x - 1$$

The following is a plot of this quartic function:

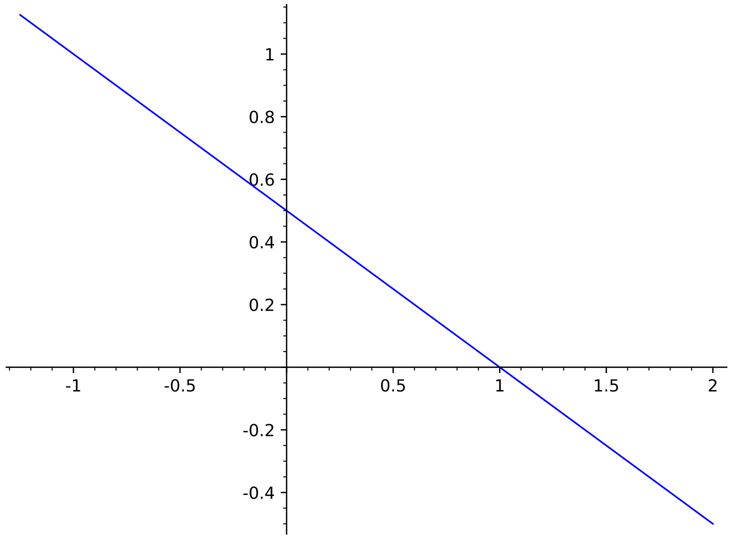


As the reader can see, the polynomial function crosses the x-axis at (x=-1), but at (x=+1) two different behaviors happen:
At (x=+1), the curve reaches the x-axis, and forms a tangent with it, and additionally, the curve ends up on the opposite side of the x-axis. This happens at only one exact value for the parameter: (+1).
This curve can be manipulated to have slightly more negative slope, exactly at x = (+1), by adding a linear function of (x) to it...

```
L(x) = -(1/2)*(x - 1)
```

$$-\frac{1}{2}\,x + \frac{1}{2}$$

The following is the plot of the linear function:

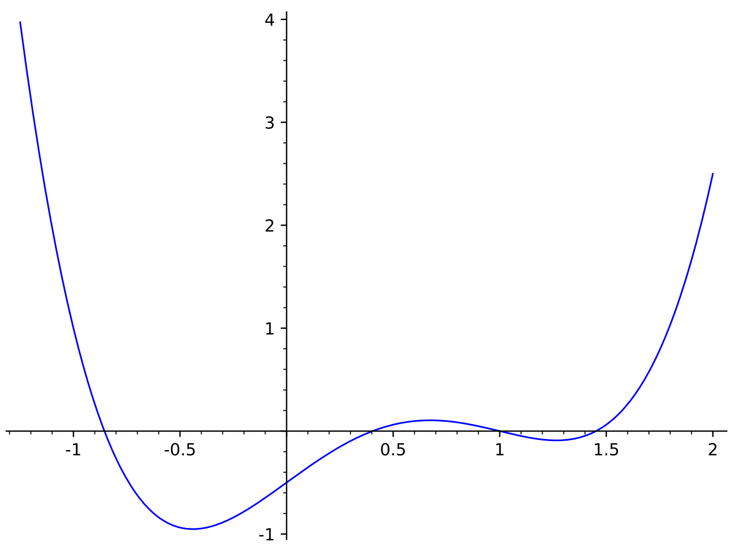


Now, a modified polynomial can be written, which is the sum of the two curves above...

```
G(x) = expand(F(x) + L(x))
```

$$x^4 - 2\,x^3 + \frac{3}{2}\,x - \frac{1}{2}$$

The following is the plot of the polynomial, which has only been modified slightly:



Examining the region around x = (+1) reveals, that the curve now only has one of its previous two behaviors: It crosses the x-axis ... But, it now does so 3 times in place of of once!

And, in the search for exactitudes, ”SageMath” can now be asked for an approximation of the 4 roots that result:

```
z = G(x).roots(x, ring=CC, multiplicities=False)
```

$$[-0.854637679718461, 0.403031716762685, 1.00000000000000, 1.45160596295578]$$

The question can next be asked, how the presence of doubled roots, or higher multiplicities can be detected, by a computer program, given a polynomial.

What was observed in the case of doubled roots, was simply that at a real root, the slope of the curve was also zero, for which reason a tangent with the x-axis was able to form.

The slope of a curve corresponds to the derivative of its function, which means that this derivative will also have a root, at the same value of (x).

Well the derivative of a quartic is a cubic,
the derivative of a cubic is a quadratic,
and the derivative of a quadratic is a
linear function, the derivative of which is
a constant.

This is a cubic:

```
Fd(x) = diff(F(x), x, 1)
```

$$4\,x^3 - 6\,x^2 + 2$$

This is a quadratic:

```
Fdd(x) = diff(F(x), x, 2)
```

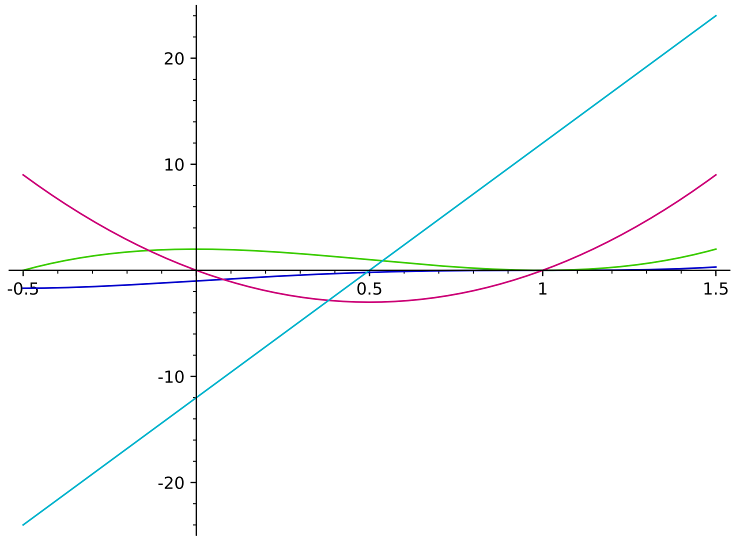
$$12\,x^2 - 12\,x$$

This is a linear function:

```
Fddd(x) = diff(F(x), x, 3)
```

$$24\,x - 12$$

This is how the series of derivatives plots:



It can be seen that one root of the quartic coincides with one root of its first, as well as of its second derivative, that being (+1), but that the third derivative (the linear function) has no corresponding root.

It therefore also follows that the first derivative, itself a cubic, has a doubled root at (x=+1), because the second derivative, the quadratic, has a plain root there.

A numerical tool has been written, which follows suit by computing all the derivatives of an original polynomial, and which also 'normalizes' them, so that the coefficient of the highest-exponent term in each was made equal to (+1) again, in order to spot doubled roots from the resulting linear equation up, so that at no point the search algorithm for roots, needs to find them if they are doubled.

A known root of a derivative was tested against all the parent equations - as resulting in close to zero or not - and if this parameter-value did produce (y=0) again, then it was factorized out of the parent equations, before the search algorithm was run on the remaining quotients.

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