Two Examples of Improper Integrals (PDF Version)

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In Calculus 2, Students are introduced to the concept of integrals, which are also called "anti-derivatives", but also to the concepts of infinite series, which may converge or may not, and to the concept of "Improper Integrals", which are similar to infinite series in their properties. In short, improper integrals are integrals, which involve infinity in some way. They are generally written in the notation of definite integrals, and are often recognizable in two ways:

- The upper end of the interval could be infinity,
- The lower end of the interval could be zero, but the integral could be of a function, which is infinite when its parameter is actually zero.

Just as infinite series exist which do not converge, Students are mainly given exercises to solve, in which the improper integral is solvable, even though when such examples are found 'in the wild', they may be unsolvable more often.

The following integral is solvable:

$$y = \lim_{b \to \infty} \int_1^b e^{-x} dx$$

$$y = \lim_{b \to \infty} \left[-e^{-x}\right]_1^b = \left(0 - (-e^{-1})\right) = e^{-1}$$

The following integral is also solvable:

$$y = \lim_{a \to 0^+} \int_a^1 x^{-\frac{1}{2}} dx$$

$$x^{-\frac{1}{2}} \equiv \frac{1}{\sqrt{x}}$$

$$y = \lim_{a \to 0^+} \left[2x^{+\frac{1}{2}}\right]_a^1 = \lim_{a \to 0^+} \left[2\sqrt{x}\right]_a^1 = (2 - 0) = 2$$

A bit of a philosophical question could be asked, of whether improper integrals are in fact definite integrals. What is happening is that the original function is not defined at the endpoints of the interval, while the integral is. Indefinite integrals are cases in which the endpoints are not defined, but where the solution has an arbitrary constant, which must be written, and which corresponds to the integral of the lower endpoint, which has not been stated. Therefore, the indefinite integral's arbitrary constant is also undefined. As long as nobody has stated the endpoints of the indefinite integral, it is not an improper one, for which reason the improper integrals are indeed definite integrals.

What is happening:

This is a plot, of the second function which I integrated above, such that the original function was undefined at one endpoint:

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F(x) = 1/sqrt(x)
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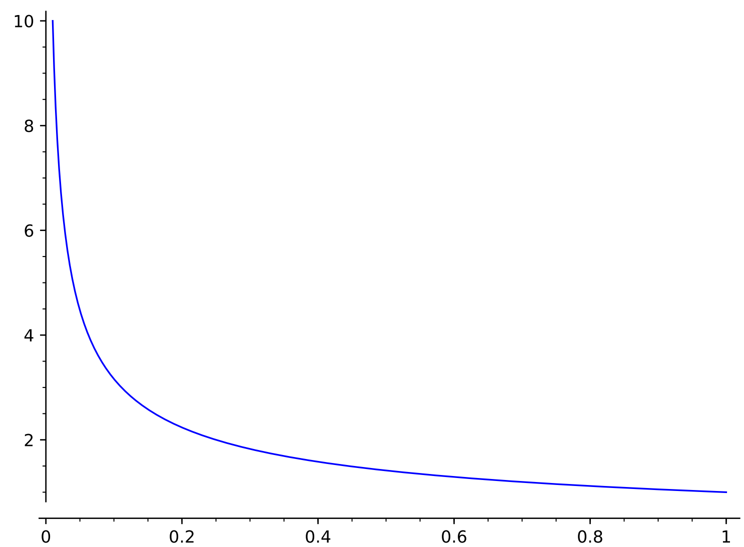
$$\frac{1}{\sqrt{x}}$$



It can happen that as (x) approaches zero, (y) approaches infinity, but that the curve of the function hugs the Y-axis so closely, that the area between $(x = 0)$ and $(x = 1)$ remains finite. Stated differently, instead of the curve approaching the X-axis closely, it can happen that it approaches the Y-axis so closely, as to give the same result.

<u>An example which will not work</u>:

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G(x) = 1/x
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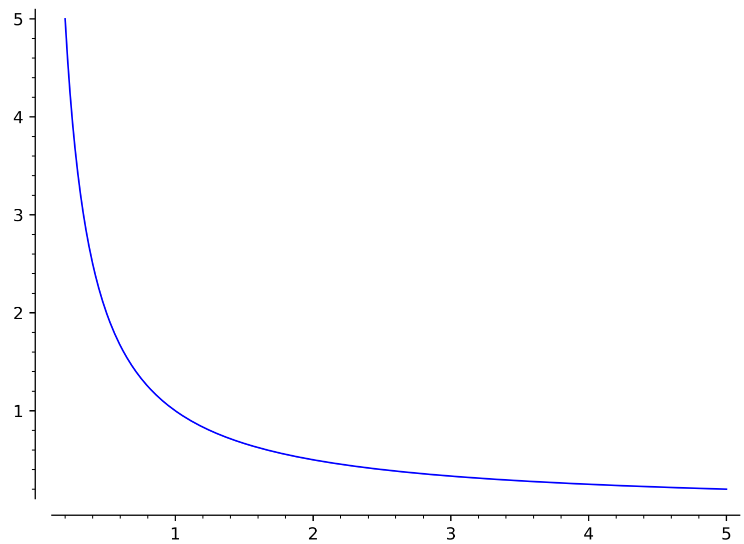
$$\frac{1}{x}$$



The integral of $(G(x))$ would be:

$y = \int x^{-1} dx = \ln |x| + C$

Because the logarithm of zero is undefined, the reciprocal function will not behave correctly, when a definite integral is to be computed, with zero anywhere in its interval. Equally, the apparent logarithm of infinity, would be infinity. Whether the definite integral has proper solutions can only be decided, by actually finding the indefinite integral analytically, and then seeing whether the resulting function is in fact defined over the stated interval.

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