The purpose of this worksheet is, to explore the question
of whether 'tripled roots' of polynomials are,
in fact, a real possibility, which would also be known
as a 'multiplicity of 3'.

The most trivial example might be:
xˆ3 = 0

But, because we cannot see the forest for the trees,
I'll just assume that this example is insufficient.

In order to start, a quartic polynomial can be
constructed, which has (+1) with a mutliplitiy
of 3, but which only has (-1) as a single real root...

--> F(x) := expand((x - 1)*(x - 1)*(x - 1)*(x + 1));

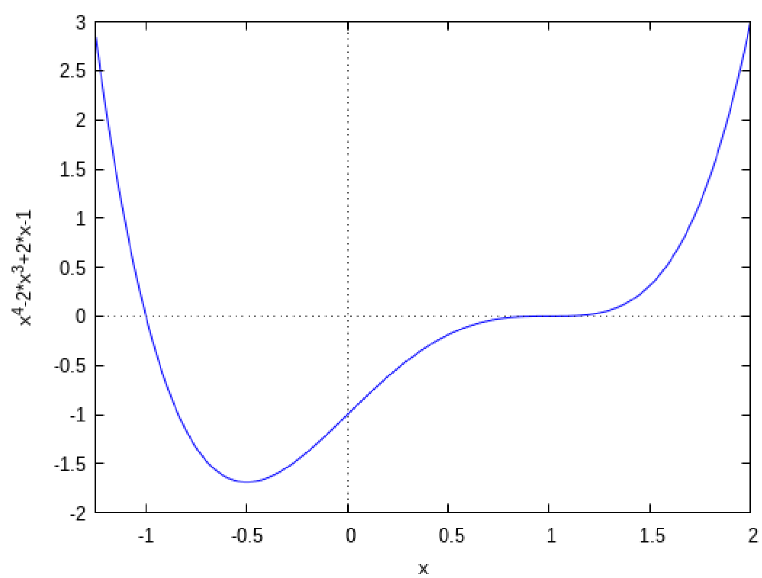
$$\mathrm{F}(x) := \mathrm{expand}\left((x-1)\ (x-1)\ (x-1)\ (x+1)\right) \qquad (\%o1)$$

--> F(x);

$$x^4 - 2x^3 + 2x - 1 \qquad (\%o2)$$

The following is a plot of this quartic function:

--> wxplot2d([F(x)], [x,-1.25,2])$



(%t3)

As the reader can see, the polynomial function crosses the x-axis at (x=-1), but at (x=+1) two different behaviors happen:

At (x=+1), the curve reaches the x-axis, and forms a tangent with it, and additionally, the curve ends up on the opposite side of the x-axis. This happens at only one exact value for the parameter: (+1).

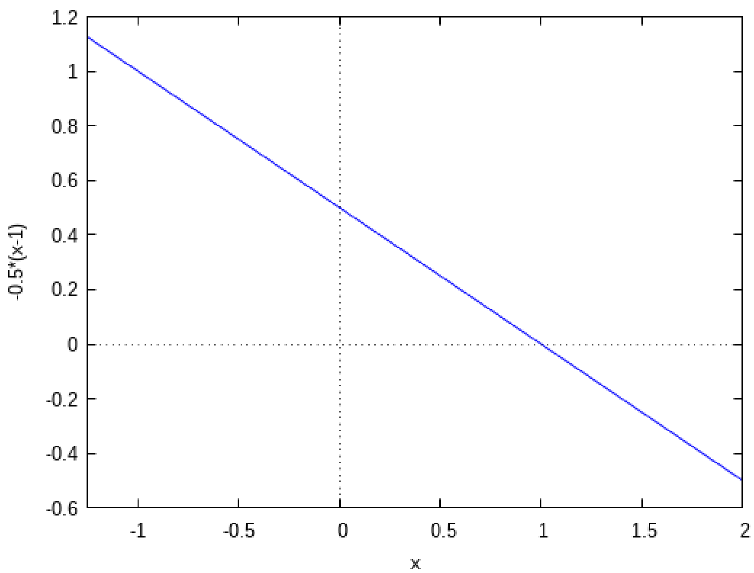
This curve can be manipulated to have slightly more negative slope, exactly at x = (+1), by adding a linear function of (x) to it...

--> L(x) := -0.5*(x - 1);

$$\mathrm{L}(x) := (-0.5)\ (x-1) \tag{\%o4}$$

The following is the plot of the linear function:

--> wxplot2d([L(x)], [x,-1.25,2])$



(%t5)

Now, a modified polynomial can be written, which is the sum of the two curves above...

--> G(x) := expand(F(x) + L(x));

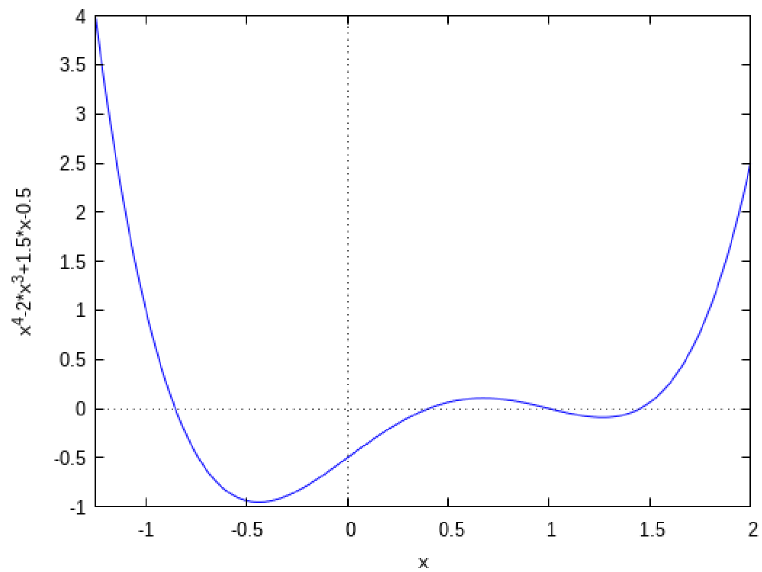
$$\mathrm{G}(x) := \mathrm{expand}\left(\mathrm{F}(x) + \mathrm{L}(x)\right) \tag{\%o6}$$

–> G(x);

$$x^4 - 2x^3 + 1.5x - 0.5 \quad (\%o7)$$

The following is the plot of the polynomial,
which has only been modified slightly:

–> wxplot2d([G(x)], [x,-1.25,2])$



(%t8)

Examining the region around x = (+1) reveals,
that the curve now only has one of its previous
two behaviors: It crosses the x-axis ... But,
it now does so 3 times in place of of once!

And, in the search for exactitudes, "Maxima" can now
be asked for an approximation of the 4 roots that result:

–> fpprintprec: 7$

–> float(realroots(G(x)));

$$[x = -0.8546376, x = 0.4030317, x = 1.0, x = 1.451605] \quad (\%o10)$$

The question can next be asked,
how the presence of doubled roots, or
higher multiplicities can be detected, by

a computer program, given a polynomial.

What was observed in the case of doubled roots,
was simply that at a real root, the slope of
the curve was also zero, for which reason a tangent
with the x-axis was able to form.

The slope of a curve corresponds to the
derivative of its function, which means that
this derivative will also have a root, at the
same value of (x).

Well the derivative of a quartic is a cubic,
the derivative of a cubic is a quadratic,
and the derivative of a quadratic is a
linear function, the derivative of which is
a constant.

This is a cubic:

−−> Fd(x):=diff(F(x), x, 1);

$$\mathrm{Fd}(x) := \frac{d}{dx}\,\mathrm{F}(x) \qquad (\%o11)$$

−−> Fd(x);

$$4x^3 - 6x^2 + 2 \qquad (\%o12)$$

This is a quadratic:

−−> Fdd(x):=diff(F(x), x, 2);

$$\mathrm{Fdd}(x) := \frac{d^2}{dx^2}\,\mathrm{F}(x) \qquad (\%o13)$$

−−> Fdd(x);

$$12x^2 - 12x \qquad (\%o14)$$

This is a linear function:

−−> Fddd(x):=diff(F(x), x, 3);

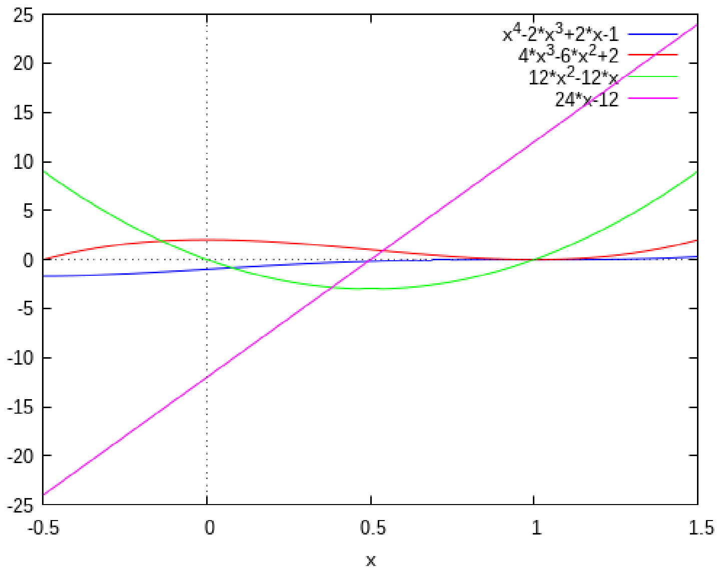
$$\mathrm{Fddd}(x) := \frac{d^3}{dx^3}\,\mathrm{F}(x) \qquad (\%o15)$$

−> Fddd(x);

$$24x - 12 \quad (\%o16)$$

This is how the series of derivatives plots:

−> wxplot2d([F(x),Fd(x),Fdd(x),Fddd(x)], [x,-0.5,1.5])$



(%t17)

It can be seen that one root of the quartic coincides with one root of its first, as well as of its second derivative, that being (+1), but that the third derivative (the linear function) has no corresponding root.

It therefore also follows that the first derivative, itself a cubic, has a doubled root at (x=+1), because the second derivative, the quadratic, has a plain root there.

A numerical tool has been written, which follows suit by computing all the derivatives of an original polynomial, and which also 'normalizes' them, so that the coefficient of the highest-exponent term in each was made equal to (+1) again, in order to spot doubled roots from the resulting linear equation up, so that at no point the search algorithm for roots, needs to find them if they are doubled.

A known root of a derivative was tested against all the parent equations - as resulting in close to zero or not - and if this parameter-value did produce (y=0) again, then it was factorized out of the parent equations, before the search algorithm was run on the remaining quotients.

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