This page will describe how the value of Pi can be computed, using the Taylor Series expansion of the arc sine function.

$\sin(\frac{\pi}{6}) = \sin(30°) = \frac{1}{2}$

$\frac{\pi}{6} = \arcsin(\frac{1}{2})$

$\pi = 6\ \arcsin(\frac{1}{2})$

From elsewhere, it can be found that the Taylor Series expansion of:

$\arcsin(x) = (x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\frac{3}{4}\frac{x^5}{5} + \frac{1}{2}\frac{3}{4}\frac{5}{6}\frac{x^7}{7} + \cdots)$

**Observation 1**: When (x) is close to (1) the above series does not converge. This is to be expected, because the slope of arcsin(1) is infinite, and Taylor Series are based on nth-order derivatives of the function in question, when (x=0). But, when (x=1/2), the series converges.

**Observation 2**: This method requires that a number of terms be computed systematically, that approaches infinity, but that each term in the resulting summation becomes infinitesimal. Even though Pi is agreed to be a Real Number, any method actually to compute it to an arbitrary level of accuracy, will make similar assumptions. Yet, Infinity and Infinitesimal are not part of the set of Real Numbers. This leads to the conclusion, that the mundane definition of Real Numbers, as being much more than Rational Numbers, is incomplete by itself, until it's extended into something more special, such as maybe the set of Hyperreal Numbers, which in turn, is an abstract concept.

**Observation 3**: The form in which I expressed the Taylor Series, specifically for the arc sine function, is the same as this form for expressing the Taylor Series for any function, which is infinitely differentiable:

$F(x) = (F(0) + \frac{1}{1!}F'(0)x + \frac{1}{2!}F''(0)x^2 + \frac{1}{3!}F'''(0)x^3 + \frac{1}{4!}F''''(0)x^4 + \frac{1}{5!}F'''''(0)x^5 + \cdots)$

This form assumes that the function and its derivatives have values at (x=0) which are easily knowable, without having to use the Taylor Series itself to compute them. There is an alternative form of the Taylor Series, which assumes that the function and all its derivatives are known not at (x=0) but at (x=a). In this example, there is no need to invoke that form.

It's trivially possible, for the evenly-numbered derivatives, or the odd-numbered ones, to be non-trivial expressions which happen to be zero, when the parameter is zero. If all the evenly-numbered terms are zero, then the function is symmetric with respect to (+/- x).

Unfortunately, this form also assumes that the reader can recognize the pattern of the terms which are to be summed, so that the reader can "Induce" the entire series. For certain purposes, the person writing the expression may need to take such work off the hands of the reader. One way to take the guesswork out of such an exercise, would be to use "Sigma Notation" - named after the symbol to be used being named "Sigma" - and for the Taylor Series in general, would be:

$F(x) = \sum_{n=0}^{\infty} \frac{1}{n!}F^n(0)x^n$

Unfortunately, for me to be able to write the Taylor Series of the arc sine function using Sigma Notation, I would also need to be able to find the nth-order derivatives of the arc sine function, myself. This is currently beyond my ability, and so I need to use the slightly inferior form at the top of this page.

**Observation 4**: The reason for which the (n!) term does not appear as a divisor, in the terms of the summation, that is the Taylor Series for the arc sine function, is also the reason for which at and above (x=1), this series diverges. This is because, as we differentiate the arc sine function multiple times, we obtain functions *with increasing amplitude*, which, when divided by (n!), leave the terms visible in the summation at the top of this worksheet. If the (1 / n!) term was still 'left over' in the summation, then, this method of deriving answers would potentially also work for values of (x > 1). The series would converge 'better'.

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