Defining an inverse trig function, to accept complex arguments.

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Normally, one would expect that an inverse trig function would only allow a real number from [-1.0 .. +1.0] as argument, and return a real number from [0.0 .. (2π)). However, this article will try to show how an inverse trig function can be defined to accept a complex argument as input, yet still, only be able to output a real number.

$\theta_1 = \arccos(z)$
$\theta_1 = \frac{1}{i}\ln\left(z \pm i\sqrt{1-z^2}\right)$

$\theta_2 = \arcsin(z)$
$\theta_2 = \frac{1}{i}\ln\left(\pm\sqrt{1-z^2} + zi\right)$

$z_1 = \sin(\theta)$
$z_1 = (\frac{1}{2i})\left(e^{\theta i} - e^{-\theta i}\right)$

$z_2 = \cos(\theta)$
$z_2 = (\frac{1}{2})\left(e^{\theta i} + e^{-\theta i}\right)$

$\theta_3 = \arctan(z)$
$\theta_3 = \frac{1}{i}\ln\left(\frac{1+zi}{\sqrt{1+z^2}}\right)$

$= \frac{1}{2i}\ln\left(\frac{(1+zi)^2}{1+z^2}\right)$
$= \frac{1}{2i}\ln\left(\frac{(1+zi)^2}{(1+zi)(1-zi)}\right)$
$= \frac{1}{2i}\ln\left(\frac{1+zi}{1-zi}\right)$
$= \frac{1}{2i}\ln\left(\frac{i-z}{i+z}\right)$

$\theta_4 = \cot^{-1}(z)$
$\theta_4 = \frac{1}{i}\ln\left(\frac{z+i}{\sqrt{z^2+1}}\right)$

$= \frac{1}{2i}\ln\left(\frac{(z+i)^2}{z^2+1}\right)$
$= \frac{1}{2i}\ln\left(\frac{(z+i)^2}{(z+i)(z-i)}\right)$
$= \frac{1}{2i}\ln\left(\frac{z+i}{z-i}\right)$

Addendum September 25, 2020:

There exists a basic, underlying fact about the imaginary constant, which most people already know, who have studied it, but which should be mentioned here for readers who may not have. The following equation is thought to be a tautology:

$\frac{1}{i} \equiv -i$

Therefore, the following two equations have exactly the same meaning:
$\arctan(z) \equiv \frac{1}{2i} \ln\left(\frac{i-z}{i+z}\right)$, $\arctan(z) \equiv -\frac{i}{2} \ln\left(\frac{i-z}{i+z}\right)$

Such facts can explain the observation that, according to common sense, 'The hyperbolic arctangent of (+1) approaches ($+\infty$), even though in some cases, the arctangent of an imaginary number (w), has been defined as (i) times the hyperbolic arctangent of ($\frac{w}{i}$).' The logarithm above approaches (-∞), while an attempt is made to compute the logarithm of (0).

In a similar vein, the following is thought to be true:

$\ln\frac{a}{b} = -\ln\frac{b}{a}$,
$\therefore -\frac{i}{2} \ln\left(\frac{i-z}{i+z}\right) \equiv \frac{i}{2} \ln\left(\frac{i+z}{i-z}\right) \equiv i\ \tanh^{-1}\left(\frac{z}{i}\right) \ldots$

Done.